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THE PERTURBED GALERKIN METHOD AND THE GENERAL THEORY FOR NON-LINEAR EQUATIONS

G.M. Vaynikko

ABSTRACT. This article deals with the application of L.V. Kantorovich's general theory of approximate methods for the solution of non-linear equations.

The general theory of approximate methods developed by L.V. Kantorovich [1], [2] plays a great role in linear problems. In this article an attempt is made to construct an analogous theory for nonlinear equations. /723*

¶1. Introduction

The approximate methods considered in [1], [2] can be interpreted as the perturbed method of Galerkin. Let us clarify this on the basis of an example of an equation of the second order:

$$x = Tx + f \tag{1.1}$$

where T is a continuous linear operator in the Banach space E, and the operator I - T continously reversible. The solutions of the following equation are adopted as approximate solutions of (1.1):

$$x_n = T_n x_n + P_n f, \tag{1.2}$$

where T_n is a continuous linear operator in enclosed sub-space E_n of space E, and P_n is a continuous linear operator of a projection onto the sub-space E_n. Spaces E and E_n and operators T and T_n are related by the conditions:

1) for any $z_n \in E_n$

$$\|P_n T z_n - T_n z_n\| \leqslant \eta_n \|z_n\|;$$

2) for any $x \in E$ there is such a $z_n \in E_n$ that

$$\|Tx - z_n\| \leqslant \eta_n' \|x\|;$$

3) at a certain $f_n \in E_n$

$$\|f - f_n\| \leqslant \eta_n'' \|f\|.$$

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Assuming conditions

$$\eta_n,\ \|P_n\|\eta_n',\ \|P_n\|\eta_n'' \to 0 \quad \text{when} \quad n \to \infty, \tag{1.3}$$

to be satisfied in (1), (2) there is proven convergence of the approximate

*)Numbers in margin indicate pagination in foreign text

solutions to an exact one *).

We shall write equations (1,2) as

$$x_n = P_n T x_n + S_n x_n + P_n f, \tag{1.2'}$$

where $S_n = T_n - P_n T$ is a linear operator in subspace E_n. When $S_n = 0$, the equation (1.2') is an (abstract) Galerkin method of solving equation (1.1); in the general case (1.2) can naturally be called the perturbed Galerkin method. Condition 1) signifies that $\|S_n\| \leqslant \eta_n$, and together with (1.3) this has, as a consequence, the relationship

$$\|S_n\| \to 0 \quad \text{when} \quad n \to \infty. \tag{1.4}$$

It is not difficult to see that in view of conditions 2), 3) and (1.3) we have (compared with remark 2 for theorem 1 below)

$$\|T - P_n T\| \to 0, \qquad \|f - P_n f\| \to 0 \text{ when } \qquad n \to \infty. \tag{1.5}$$

We can verify that conditions (1.4) and (1.5) by themselves guarantee the convergence of the approximate solution to an exact one **). This assumption, 1) - 3) and (1.3) may be replaced by less restrictive conditions (1.4) and (1.5). This makes it possible in particular to do away with restrictidness of projection operator P_n. The necessity for such a generalization arises, for example, when the collocation method is investigated (See (3)).

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*) In [1], [2] only subspace E_n is assumed to be complete and E may be an incomplete linear standardized space. Let us note that if conditions 1) - 3) have been satisfied in the case of an incomplete E, these conditions also remain satisfied for the completion of E and operator T which is extended with respect to continuity. Therefore, from the theoretical standpoint the assumption concerning the completeness of E does not restrict the generality.

**) This fact can be easily proved directly. It can, however, be obtained from theorem 3 (see below) by writing equation (1.1) in the form $x = \Gamma x$ with nonlinear operator $\Gamma x = Tx + f$ (in the formation of theorem 3, T must be replaced by Γ).

The above remarks will be taken into consideration when nonlinear equations are investigated. The analog of approximate equation (1.2) will be immediately given in the form of (1.2') and the convergence conditions will be formulated in terms analogous to (1.4) and (1.5)

¶ 2. The Perturbed Galerkin Method for Nonlinear Equations

In Bonach space E let us consider equation

$$x = Tx \tag{2.1}$$

with a nonlinear operator T which is continuous on an open set $\Omega \subset E$.

Let $\{E_n\}$ be a sequence of closed subspace in E, and let P_n be a linear operator of projection into space E_n, i.e., $P_n z \in E_n$ for any value z from the domain of operator P_n and $P_n z_n = z_n$ for $z_n \in E_n$. Operator P_n can also be unbounded, and in such a case it is assumed that the range of values of T (Ω) of operator T on Ω is included in the domain of P_n, and that operator $P_n T$ is also continuous on Ω. /725

As approximate solution of equation (2.1) adopted are the solutions of equations

$$x_n = P_n T x_n + S_n x_n \tag{2.2}$$

(the perturbed Galerkin method), where S_n is a generally nonlinear operator in space E_n, continuous on the set $\Omega_n = \Omega \cap E_n$. Equation (2.2) is considered in E_n.

Below will be given two proofs of convergence of the perturbed Galerkin method (2.2). The first proof makes use of the concpet of the rotation of completely continuous vector field (see (4)), and accordingly, operator T, $P_n T$ and S_n are assumed to be completely continuous. The second proof is based on the principle of compressed mappings, and instead of complete continuity there is introduced the assumption of operator differentiability.

The obtained results are transferred in the usual manner to the case where the approximate equation is given not for subspace E_n, but for some other Banach space $\bar{E}_n$, which is isomorphic to E_n (see (1),(2)).

For the sake of simplifying the formulation, we shall limit ourselves only to equation (2.2) in subspace $E_n \subset E$. Even though in the application the exact equation and approximate equation are, as a rule, given in different space; it is usually not difficult to write an equivalent equation in the form of (2.1) and (2.2).

We shall henceforth everywhere adhere to the notation: I is an identity operator, $p^{(n)} = I - P_n$, $\rho(x, \mathfrak{M})$ is the distance of point $x \in E$ to set $\mathfrak{M} \subset E$.

¶3. The first Proof of Convergence

In this section we shall assume that the set $\Omega \subset E$ is bounded, operator T is determined at closing of $\bar{\Omega} = \Omega \cup \Gamma$ and does not have at the boundary Γ the set Ω of fixed points. For the sake of simplicity let us also assume that $\Gamma_n = \Gamma \cap \Omega_n$ is for each value of n the boundary of the set $\Omega_n = \Omega \cap E_n$ in E_n.

Let us denote by X_o $(X_o \subset \Omega)$ a closed set consisting of the fixed points of operator T in Ω (set of solutions of equation (2.1)). Set X_o is not empty if operator T is entirely continuous on $\bar{\Omega}$ and the rotation of the vector field x - Tx on Γ differs from zero: $\gamma(I - T; \Gamma) \neq 0$. Let us denote by $X_n (X_n \subset \Omega_n)$ the set of solutions of equation (2.2) in Ω_n.

Theorem 1. Let operators T and $P_n T$ be entirely continuous in the set $\bar{\Omega} = \Omega \cup \Gamma$ and operator S_n entirely continuous on the set $\bar{\Omega}_n = \Omega_n \cup \Gamma_n$, and furtheremore:

$$\sup_{x \in \bar{\Omega}} \|P^{(n)} Tx\| \to 0, \quad \sup_{x \in \bar{\Omega}_n} \|S_n x_n\| \to 0 \quad \text{when} \quad n \to \infty. \tag{3.1}$$

Then if $\gamma(I - T; \Gamma) \neq 0$, with sufficiently large values of n the set X_n of the solutions of equation (2.2) is not empty, and /726

$$\sup_{x_n \in X_n} \rho(x_n, X_0) \to 0 \quad \text{when} \quad n \to \infty. \tag{3.2}$$

Proof. By contradition, it is easy to prove the existence of such a constant $\alpha > 0$ such that

$$\|x - Tx\| \geqslant \alpha \qquad (x \in \Gamma) \tag{3.3}$$

(recollect that X_o does not contain points of Γ). By virtue of the first of the conditions of (3.1), with sufficiently large values of n (let this be when $n \geqslant n_0$), we have

$$\|P^{(n)} Tx\| \leqslant \frac{\alpha}{2} \qquad (x \in \Gamma). \tag{3.4}$$

From conditions (3.3) and (3.4) it follows that vector fields x - Tx and x - P_nTx are isoniotopic on T, and therefore

$$\gamma(I - P_n T;\ \Gamma) = \gamma(I - T;\ \Gamma) \neq 0 \qquad (n \geqslant n_0).$$

Further, from the definition of the rotation of a vector field it follows that with sufficiently large values of n (let this be when $n \geqslant n_1, n_1 \geqslant n_0$)

$$\gamma(I - P_n T;\ \Gamma) = \gamma(I - P_n T;\ \Gamma_n)$$

(in the right-hand part of the equation I - P_nT is regarded as an operator in E_n). Therefore,

$$\gamma(I - P_n T; \Gamma_n) \neq 0 \quad (n \geqslant n_1).$$

According to (3.3) and (3.4) we obtain

$$\|x-P_nTx\| \geqslant \frac{\alpha}{2} \quad (x \in \Gamma_n;\ n \geqslant n_1). \tag{3.6}$$

By virtue of the second condition (3.1), when $n \geqslant n_2$ $(n_2 \geqslant n_1)$

$$\|S_nx_n\| \leqslant \frac{\alpha}{4} \quad (x \in \Gamma_n). \tag{3.7}$$

From inequatities (3.6) and (3.7) we conclude that vector fields $x - P_nT_x$ and $x - P_nTx - S_nx$ are homotopic on T_n, therefore

$$\gamma(I-P_nT-S_n;\ \Gamma_n) = \gamma(I-P_nT;\ \Gamma_n) \qquad (n \geqslant n_2)$$

and (see(3.5)) $\gamma(I-P_nT-S_n;\ \Gamma_n) \neq 0$. Therefore the operator $I - P_nT - S_n$ has in $_n$at least one fixed point, i.e., when $n \geqslant n_2$ set X_n is not empty.

Let us prove relationship (3.2). Let it be given that $\varepsilon > 0$, and let us surround each point $x_0 \in X_0$ by an open sphere with a radius not greater than ε and with the center in x_0, entirely contained in Ω. We shall denote the thus obtained converging of set X_0 by X_0^{ε}. It is clear that in closed set $\bar{\Omega} \setminus X_0$ there are no fixed points of operator T. Let us take such a number $\alpha_\varepsilon > 0$ that

$$\|x-Tx\| \geqslant \alpha_\varepsilon \quad (x \in \bar{\Omega} \setminus X_0^{\varepsilon}).$$

By virtue of conditions (3.1), with sufficiently large values of n (when $n \geqslant n_\varepsilon$) /727

$$\|x-P_nT_x-S_nx\| \geqslant \frac{\alpha_\varepsilon}{2} \quad (x \in E_n \cap (\bar{\Omega} \setminus X_0^{\varepsilon})).$$

Thus, when $n \geqslant n_\varepsilon$ not a single point of the set X_n can be situated in $\bar{\Omega} \setminus X_0^{\varepsilon}$, i.e., $X_n \subset X_0^{\varepsilon}$. In view of the arbitrariness of $\varepsilon > 0$ this is equivalent to (3.2). Theorem 1 has been proven.

The proof carried out is a variation of the reasoning used for (4) in proving the convergence of the Galerkin method.

Corrolary. Let the conditions of theorem 1 be satisfied. Then if equation (2.1) has an isolated solution $x_0 \in \Omega$ with a nonzero index, then such values of $\sigma_0 > 0$ and n_0, can be found, that when $n \geq n_0$ equation (2.2) has in sphere $\|x-x_0\| \leqslant \delta_0$ at least one solution x_n. The correspondence $\|x_n - x_0\| \to 0$ takes place when $n \to \infty$.

This confirmation we obtain from theorem 1, taking as a "new" set Ω the sphere $\|x-x_0\| < \delta_0$ of a sufficiently small radius σ_0. Let us note that the uniqueness of approximation x_n may, with a sufficiently large value of n, be violated in any sphere $\|x-x_0\| \leqslant \delta$.

Let us point out certain insufficient conditions for the satisfaction of the first of relationships (3.1).

Remarks: 1. Let Banach space $E' \subset E$ be continuously inserted into E. Let operator T transfer $\bar{\Omega}$ into a subset which is compact in E', and let projection operators P_n $(n = 1, 2, \ldots)$ be bounded as operators from E into E and let them tend strongly toward the operator of the insertion E into E. Then

$$\sup_{x \in \bar{\Omega}} \|P^{(n)} Tx\| \to 0 \quad \text{when} \quad n \to \infty. \tag{3.8}$$

The proof can be easily carried out by contradiction see (3).

2. Let for every $x \in \bar{\Omega}$

$$\rho(Tx, E_n) \leqslant \eta_n \quad (n = 1, 2, \ldots).$$

Let the operators P_n $(n = 1, 2, \ldots)$ be limited in E and

$$\|P_n\| \eta_n \to 0 \quad \text{when} \quad n \to \infty.$$

Then (3.8) takes place. (Operator T does not have to be entirely continuous.)

Actually, since $P^{(n)} z_n = 0$ for $z_n \in E_n$, for any $x \in \bar{\Omega}$ we have

$$\|P^{(n)} Tx\| \leqslant \|P^{(n)}\| \rho(Tx, E_n) \leqslant (1 + \|P_n\|) \eta_n \to 0 \quad \text{when} \quad n \to \infty.$$

¶ 4. Concerning the Solubility of the Nonlinear Equations

Recollect that operator T, determined on the set Ω is called differentiable operators according to Frechet in point $x_0 \in \Omega$, if with any sufficiently small, with respect to the norm, element $h \in E$ (such that $x_0 + h \in \Omega$) , the increase of $T(x_0 + h) - Tx_0$ can be introduced in the form of

$$T(x_0 + h) - Tx_0 = T'(x_0) h + \omega(x_0; h), \tag{4.1}$$

where $T'(x_0)$ is a linear continuous operator in E (a Frechet derivative of operator T in point x_0) , and $\|\omega(x_0; h)\| / \|h\| \to 0$ when $\|h\| \to 0$.

If T is differentiable in every point of the segment connecting points x_0 and $x_0 + h$, and U is any continuous linear operator in E, then see (2). /728

$$\|T(x_0 + h) - Tx_0 - Uh\| \leqslant \|h\| \sup_{0 < \theta < 1} \|T'(x_0 + \theta h) - U\|. \tag{4.2}$$

In particular, assuming $U = T'(x_0)$, we have

$$\|\omega(x_0; h)\| = \|T(x_0 + h) - Tx_0 - T'(x_0) h\| \leqslant \tag{4.3}$$

$$\leqslant \|h\| \sup_{0 < \theta < 1} \|T'(x_0 + \theta h) - T'(x_0)\|.$$

Operator T is called continuously differentiable in point x_0, if it is differentiable in every point of a certain vicinity of point x_0, and furthermore $\|T'(x) - T'(x_0)\| \to 0$ when $n \to \infty$.

Let T and $\tilde{T}$ be continuous operators on the open set $\Omega \subset E$. Let us determine the conditions under which the solubility of one of the equations, $x = Tx$ and $x = \tilde{T}x$ leads to the solubility of the other.

Theorem 2. Let the equation $x = \tilde{T}x$ have the solution $\tilde{x}_0 \in \Omega$, and let the following conditions be satisfied:

a) operator T is differentiable according to Frechet in certain vicinity of point $\tilde{x}_0$, and furthermore linear operator $I - T'(\tilde{x}_0)$ is continuously invertible;

b) at certain values of σ and $(\delta > 0, 0 \leqslant q < 1)$ the inequalities

$$\sup_{\|x-\tilde{x}_0\| \leqslant \delta} \| [I - T'(\tilde{x}_0)]^{-1} [T'(x) - T'(\tilde{x}_0)] \| \leqslant q, \tag{4.4}$$

$$\alpha \equiv \| [I - T'(\tilde{x}_0)]^{-1} [T\tilde{x}_0 - \tilde{T}\tilde{x}_0] \| \leqslant \delta (1 - q). \tag{4.5}$$

are valid.*)

Then equation $x = Tx$ has in the sphere $\|x - \tilde{x}_0\| \leqslant \delta$ the unique solution x_0. The inequality

$$\frac{\alpha}{1+q} \leqslant \|x_0 - \tilde{x}_0\| \leqslant \frac{\alpha}{1-q}. \tag{4.6}$$

is valid.

Proof. From equation $x = Tx$ we subtract the equality $\tilde{x}_0 = \tilde{T}\tilde{x}_0$ term by term, and transform the result

$$x - \tilde{x}_0 = Tx - \tilde{T}\tilde{x}_0 = (Tx - T\tilde{x}_0) + (T\tilde{x}_0 - \tilde{T}\tilde{x}_0),$$
$$[I - T'(\tilde{x}_0)](x - \tilde{x}_0) = [Tx - T\tilde{x}_0 - T'(\tilde{x}_0)(x - \tilde{x}_0) + (T\tilde{x}_0 - \tilde{T}\tilde{x}_0)].$$

Applying to both parts, operator $[I - T'(\tilde{x}_0)]^{-1}$, we obtain the final result that equation $x = Tx$ is equivalent to equation $x = Ax$, where

$$Ax = \tilde{x}_0 + [I - T'(\tilde{x}_0)]^{-1} [Tx - T\tilde{x}_0 - T'(\tilde{x}_0)(x - \tilde{x}_0) + (T\tilde{x}_0 - \tilde{T}\tilde{x}_0)]$$

Let us show that A is the compression operator on the sphere $\|x - \tilde{x}_0\| \leqslant \delta$. Actually, by virtue of an analog of inequality (4.3) for operator $[I - T'(\tilde{x}_0)]^{-1}T$ when $\|x - \tilde{x}_0\| \leqslant \delta$ we have

$$\| [I - T'(\tilde{x}_0)]^{-1} [Tx - T\tilde{x}_0 - T'(\tilde{x}_0)(x - \tilde{x}_0)] \| \leqslant$$
$$\leqslant \|x - \tilde{x}_0\| \sup_{0<\theta<1} \| [I - T'(\tilde{x}_0)]^{-1} [T'(\tilde{x}_0 + \theta(x - \tilde{x}_0)) - T'(\tilde{x}_0)] \|; \tag{4.7}$$

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*) The number σ is assumed to be so small that the sphere is contained in Ω.

taking inequalities (4.4) and (4.5) into consideration, we find /729

$$\|Ax-\tilde{x}_0\| \leqslant \|x-\tilde{x}_0\| q+\alpha \leqslant \delta q+\delta(1-q)=\delta,$$

i.e., A transfers sphere $\|x-\tilde{x}_0\| \leqslant \delta$ into itself. Let x_1 and x_2 be two points of sphere $\|x-\tilde{x}_0\| \leqslant \delta.$ we have

$$Ax_1-Ax_2=[I-T'(\tilde{x}_0)]^{-1}[Tx_1-Tx_2-T'(\tilde{x}_0)(x_1-x_2)].$$

Let us write unequality (4.2) for operator $[I-T'(\tilde{x}_0)]^{-1}T$ (the role of points x_0+h and x_0 and operator U is played, respectively, by x_1, x_2 and $[I-T'(\tilde{x}_0)]^{-1}T'(\tilde{x}_0)$):

$$\|[I-T'(\tilde{x}_0)]^{-1}[Tx_1-Tx_2-T'(\tilde{x}_0)(x_1-x_2)]\| \leqslant \|x_1-x_2\| \times \\ \times \sup_{0<\theta<1} \|[I-T'(\tilde{x}_0)]^{-1}[T'(x_2+\theta(x_1-x_2))-T'(\tilde{x}_0)]\|.$$

When $0<\theta<1$, point $x_2+\theta(x_1-x_2)$ belongs to sphere $\|x-\tilde{x}_0\| \leqslant \delta$ and with aid of (4.4) we find

$$\|Ax_1-Ax_2\| \leqslant q\|x_1-x_2\|,$$

i.e., operator A compresses the sphere $\|x-\tilde{x}_0\| \leqslant \delta$.

According to the principle of compressed mappings, the equation $x=Ax$, and along with it also equation $x=Tx$, have in sphere $\|x-\tilde{x}_0\| \leqslant \delta$ the unique (general) solution x_0 . We have

$$x_0-\tilde{x}_0=Ax_0-\tilde{x}_0=[I-T'(\tilde{x}_0)]^{-1}[Tx_0-T\tilde{x}_0- \\ -T'(\tilde{x}_0)(x_0-\tilde{x}_0)+(T\tilde{x}_0-\tilde{T}\tilde{x}_0)],$$

whence, using (4.7) and (4.4), we obtain

$$\|x_0-\tilde{x}_0\| \leqslant q\|x_0-\tilde{x}_0\|+\alpha,$$
$$\|x_0-\tilde{x}_0\| \geqslant \alpha-q\|x_0-\tilde{x}_0\|.$$

The last inequalities are equivalent to estimate (4.6). Theorem 2 has been proven.

Corollary: Let operator T be continuous on an open set Ω and differentiable according to Frechet in a certain vicinity of point $x_* \in \Omega$, and furthermore let linear operator $I-T'(x_*)$ be continuously invertible. For certain values of δ and q ($\delta>0$, $0 \leqslant q<1$), let inequalities

$$\sup_{\|x-x_*\| \leqslant \delta} \|[I-T'(x_*)]^{-1}[T'(x)-T'(x_*)]\| \leqslant q, \tag{4.8}$$

$$\alpha \equiv \|[I-T'(x_*)]^{-1}(x_*-Tx_*)\| \leqslant \delta(1-q). \tag{4.9}$$

be valid.

Then equation x = Tx has in sphere $\|x-x_*\| \leqslant \delta$ this unique solution* x_0 , and the estimate is valid.

$$\frac{\alpha}{1+q} \leqslant \|x_* - x_0\| \leqslant \frac{\alpha}{1-q}. \tag{4.10}$$

Actually, $\tilde{x}_0 = x_*$ is the solution of equation x = $\tilde{T}$x, where /730

$$\tilde{T}x = T'(x_*)(x-x_*) + x_* \qquad (x \in \Omega),$$

and it can be verified without difficulty that for the equation pair, $x = Tx$ and $x = \tilde{T}x$ all the conditions of theorem 2 are satisfied.

Inequality (4.10) will be used in ¶7 in deriving estimates of the error of the perturbed Galerkin method. To establish the fact of convergence, we shall in the next section use a somewhat cruder, but more convenient formulation of theorem 2. Let us interchange the roles of operators T and $\tilde{T}$.

Theorem 2'. Let equation x = Tx have the solution $x_0 \in \Omega$, and let the following conditions be satisfied:

a) operator T is differentiable according to Fresche in a certain vicinity of point x_0, and furthermore linear operator $I - \tilde{T}'(x_0)$ is continuous invertible,

$$\|[I - \tilde{T}'(x_0)]^{-1}\| \leqslant \varkappa; \tag{4.11}$$

b) for certain values of σ and q $(\delta > 0,\ 0 \leqslant q < 1)$, inequalities

$$\sup_{\|x-x_0\| \leqslant \delta} \|T'(x) - \tilde{T}'(x_0)\| \leqslant \frac{q}{\varkappa}, \tag{4.12}$$

$$\|\tilde{T}x_0 - Tx_0\| \leqslant \frac{\delta(1-q)}{\varkappa}. \tag{4.13}$$

are valid.

The equation x = Tx has a unique solution in sphere $\|x - x_0\| \leqslant \delta$.

¶5. The Second Proof of Convergence

Theorem 3. Let operators T and P_nT be continuous on an open set $\Omega \subset E$, and let operator S_n be continuous on $\Omega_n = \Omega \cap E_n$. Let the equation (2.1) have the solution $x_0 \in \Omega$, and let the following conditions be satisfied:

* In connection with the proof of the convergence of Newton's method in (1),(2) an analogous result has been indicated under the assumption that operator T is twice differentiable and that the second derivative is bounted on sphere $\|x - x_*\| \leqslant \delta$.

1) $\|P^{(n)}x_0\| \to 0$ u $\|S_nP_nx_0\| \to 0$ when *) $n \to \infty$;

2) the operator T is continuously differentiable according to Frechet in the point x_o and linear operator $I - T'(x_0)$ is continuously invertible;

3) operator $P_n T$ is differentiable according to Frechet in a certain vicinity $\|x - x_0\| \leqslant \sigma$ of point x_o (as an operator in E), and furthermore $(P_nT)'(x) = P_nT'(x)$ and for values of $\varepsilon > 0$ such values of n_ε and σ_ε $(0 < \sigma_\varepsilon < \sigma)$ will be found, that

$$\|P^{(n)}T'(x)\| \leqslant \varepsilon \qquad \text{when} \qquad n \geqslant n_\varepsilon \ u \ \|x - x_0\| \leqslant \sigma_\varepsilon;$$

4) operator S_n is differentiable according to Frechet at points of intersection of sphere $\|x - x_0\| < \sigma$ with E_n (as an operator in E_n), and furthermore, for every value of $\varepsilon > 0$ such values of n_ε' and σ_ε' $(0 < \sigma_\varepsilon' < \sigma)$ that

$$\|S_n'(x)\| \leqslant \varepsilon \quad \text{when} \quad n \geqslant n_\varepsilon' \quad \text{and} \quad \|x - x_0\| \leqslant \sigma_\varepsilon' \qquad (x \in E_n).$$

Then such values of n_o and $\sigma_o > 0$ will be found, that the solution x_o of equation (2.1) is unique in sphere $\|x - x_0\| \leqslant \delta_0$ and when $n \geqslant n_0$ equation (2.2) has in this sphere the unique solution x_n Convergence $\|x_n - x_0\| \to 0$ occurs when $n \to \infty$, and the bilateral estimate (c_1, c_2 = const. > 0)

$$c_1 \|P^{(n)}x_0 - S_nx_n\| \leqslant \|x_n - x_0\| \leqslant c_2 \|P^{(n)}x_0 - S_nx_n\|. \tag{5.1}$$

is valid. /731

Before proceeding to the proof of the theorem, let us note some consequences of its conditions, which will be repeatedly used in the following sections.

1. From condition 3) of the theorem, it follows that

$$\|P^{(n)}T'(x_0)\| \to 0 \text{ when } \quad n \to \infty. \tag{5.2}$$

2. There exist such values of n_1, $\delta_1 > 0$ and $\varkappa = \text{const}$, that when $\|x - x_0\| \leqslant \delta_1$ and $n \geqslant n_1$, operators I - T'(x) and I - P T'(x) an invertible in E, and operator $I - P_nT'(x) - S_n'(x)$ is invertible in E_n, and furthermore

$$\|[I - T'(x)]^{-1}\| \leqslant \varkappa \qquad (\|x - x_0\| \leqslant \delta_1), \tag{5.3}$$

$$\|[I - P_nT'(x)]^{-1}\| \leqslant \varkappa \qquad (\|x - x_0\| \leqslant \delta_1; \quad n \geqslant n_1), \tag{5.4}$$

$$\|[I - P_nT'(x) - S_n'(x)]^{-1}\| \leqslant \varkappa \qquad (\|x - x_0\| \leqslant \delta_1, \quad x \in E_n; \quad n \geqslant n_1). \tag{5.5}$$

Actually, the (5.3) proceeds directly from condition 2) of the theorem,

* From $\|P^{(n)}x_0\| \to 0$ it follows that $P_nx_0 \in \Omega_n$ for sufficiently large values of n.

and (5.4) then proceeds directly from condition 3) and (5.3)*.) Estimate (5.4) remains in force, if $I - P_nT'(x)$ is regarded as an operator in E_n. Hence and from condition 4) of the theorem we obtain (5.5).

3. **For every value of $\varepsilon > 0$ there will be found such values of σ_ε and n_ε that**

$n \to \infty$,
$$\|P_nT'(x) - P_nT'(x_0)\| \leqslant \varepsilon \qquad (\|x - x_0\| \leqslant \sigma_\varepsilon; \quad n \geqslant n_\varepsilon). \tag{5.6}$$

Actually, in accordance with conditions 2) and 3) of the theorem, let us select σ_ε and n_ε in such a manner that when $\|x - x_0\| \leqslant \sigma_\varepsilon$ and $n \geqslant n_\varepsilon$ we would have

$$\|T'(x) - T'(x_0)\| \leqslant \frac{\varepsilon}{3}, \qquad \|P^{(n)}T'(x)\| \leqslant \frac{\varepsilon}{3};$$

then

$$\|P_nT'(x) - P_nT'(x_0)\| \leqslant \|P^{(n)}T'(x)\| + \|T'(x) - T'(x_0)\| + \\ + \|P^{(n)}T'(x_0)\| \leqslant \varepsilon.$$

Proof of theorem 3. The isolation of solution x_0 of equation (2.1) proceeds from the invertibility of equation $I - T'(x_0)$. In proving the solubility of equation (2.2) as an intermediate step, let us carry out the proof of this fact first for the (unperturbed) Galerkin method

$$x_n = P_nTx_n. \tag{5.7}$$

In contrast to equation (2.2), equation (5.7) may be studied both in subspace E_n and in subspace E. The solutions of equation (5.7) belong to E_n.

Let us show that for sufficiently large values of n, for equation (5.7) the conditions of theorem $2'$ $(\tilde{T} = P_nT)$ are satisfied. Condition a) is satisfied in view of condition 3) of theorem 3 and inequality (5.4) for /732
$x = x_0$. Let us show that condition b) is also satisfied. For this purpose, let these be given to a certain $q\,(0 < q < 1)$, and then let these be any sufficiently small value of $\sigma > 0$, so that with sufficiently large values of n there would occur the inequality (see (5.6))

$$\sup_{\|x - x_0\| \leqslant \delta} \|P_nT'(x) - P_nT'(x_0)\| \leqslant \frac{q}{\varkappa},$$

which corresponds to inequality (4.12). Inequality (4.13) is also valid for sufficiently large values of n, because according to condition 1) of theorem 3

$$\|\tilde{T}x_0 - Tx_0\| = \|P^{(n)}Tx_0\| = \|P^{(n)}x_0\| \to 0 \quad \text{when} \quad n \to \infty.$$

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* In (5.4) the constant $\varkappa$ will, generally speaking, be greater, and the constant σ_1, will be smaller than in (5.3), but after (5.4) has already been determined, x can be increased and σ_1 can be decreased in (5.3) in such a manner that these constants would be equal in (5.3) and (5.4). An analogous remark pertains to (5.5).

Thus, starting with a certain $n = n_\delta$, all conditions of theorem 2' are satisfied and, consequently in sphere $\|x - x_0\| \leqslant \delta$ there exists the unique solution x_n^0 of equation (5.7). Since the value $\delta > 0$ in our considerations was arbitrarily small, the convergence $\|x_n^0 - x_0\| \to 0$ when is simultaneously proved.

In order to determine the solubility of equation (2.2), let us now apply theorem 2' to equations (5.7) and (2.2). The first of these equations plays the role of an "exact" equation, and the second - plays the role of an "approximate" equation $x = \tilde{T}x$; both equations are examined in the space E_n. Since $x_n^0 \to x_0$ when $n \to \infty$, with sufficiently large values of n condition a) is satisfied:

$$\|[I - P_n T'(x_n^0)]^{-1}\| \leqslant \varkappa.$$

Let these again be given some value of q $(0 < q < 1)$ and let us then take any such sufficiently small value of $\delta > 0$, that with sufficiently large values of n we would have

$$\sup_{\|x - x_n^0\| \leqslant \delta} \|P_n T'(x) - P_n T'(x_n^0)\| \leqslant \frac{q}{3\varkappa};$$

This is possible in view of (5.6) and the convergence $\|x_n^0 - x_0\| \to 0$. Decreasing if necessary, the value σ we can in view of condition 4) of theorem 3 consider that with sufficiently large values of n is also valid.

$$\sup_{\|x - x_n^0\| \leqslant \delta} \|S_n'(x)\| \leqslant \frac{q}{3\varkappa}.$$

To sum up, we obtain this fact that with sufficiently large values of n, the analog of inequality (4.12),

$$\sup_{\|x_n - x_n^0\| \leqslant \delta} \|[P_n T'(x) + S_n'(x)] - [P_n T'(x_n^0) + S_n'(x_n^0)]\| \leqslant \frac{q}{\varkappa}. \tag{5.8}$$

is valid.

Finally, the analog of inequality (4.13)

$$\|S_n x_n^0\| \leqslant \frac{\delta(1-q)}{\varkappa} \tag{5.9}$$

is also valid with sufficiently large values of n), because from conditions 1) and 4) of the theorem it follows that

$$\|S_n x_n^0\| \leqslant \|S_n P_n x_0\| + \|S_n x_n^0 - S_n P_n x_0\| \leqslant \|S_n P_n x_0\| + \\ + \sup_{0 < \theta_n < 1} \|S_n'(x_n^0 + \theta_n (P_n x_0 - x_n^0))\| \, \|x_n^0 - P_n x_0\| \to 0 \text{ when } n \to \infty.$$

Thus all the conditions of the analog of theorem 2' are satisfied, /733
and hence the conclusion drawn that equation (2.2) has, when $n \geqslant n_\delta$ in a sphere $\|x - x_n^0\| \leqslant \delta$, the unique solution x_n. Since the value $\delta > 0$ could in our considerations be arbitrarily small, and $\|x_n^0 - x_0\| \to 0$ when $n \to \infty$, it also follows that $\|x_n - x_0\| \to 0$ when $n \to \infty$, and x_n will be the only solution of equation (2.2), also in a sphere $\|x - x_0\| \leqslant \delta_0$. The value δ_0 may be chosen so small, that in the indicated spnere there will be no other solution of equation (2.1) except x_0.

Let us determine estimate (5.1). From equality $x_n = P_nTx_n + S_nx_n$ subtract $x_0 = Tx_0$ and transform the result (see designations (4.1)).

$$x_n - x_0 = P_nTx_n - P_nTx_0 + S_nx_n - P^{(n)}x_0 =$$
$$= P_nT'(x_0)(x_n - x_0) + P_n\omega(x_0;\ x_n - x_0) + S_nx_n - P^{(n)}x_0,$$

or, denoting $y_n = P^{(n)}x_0 - S_nx_n$,

$$[I - P_nT'(x_0)](x_n - x_0) = P_n\omega(x_0;\ x_n - x_0) - y_n. \tag{5.10}$$

Using formula (4.3) for operator P_nT, we find

$$\frac{\|P_n\omega(x_0;x_n - x_0)\|}{\|x_n - x_0\|} \leqslant \sup_{0<\theta<1} \|P_nT'(x_0 + \theta(x_n - x_0)) - P_nT'(x_0)\| \leqslant \varepsilon,$$

if $n \geqslant n_\varepsilon$ and $\|x_n - x_0\| \leqslant \delta_\varepsilon$ (see 5.6)). Because $\|x_n - x_0\| \to 0$ when $n \to \infty$, the last inequality will be satisfied for sufficiently large values of n, and in view of the arbitrariness of ε

$$\frac{\|P_n\omega(x_0;x_n - x_0)\|}{\|x_n - x_0\|} \to 0 \text{ when } n \to \infty. \tag{5.11}$$

Since norms $\|I - P_nT'(x_0)\|$ and $\|[I - P_nT'(x_0)]^{-1}\|$ $(n \geqslant n_1)$ are bounded in their set (see (5.2) and (5.4)), it follows from (5.10) and (5.11) that

$$(1 - \varepsilon_n)\|[I - P_nT'(x_0)]^{-1}y_n\| \leqslant \|x_n - x_0\| \leqslant$$
$$\leqslant (1 + \varepsilon_n)\|[I - P_nT'(x_0)]^{-1}y_n\|, \tag{5.12}$$

where $\varepsilon_n \to 0$ when $n \to \infty$. Estimate (5.1) proceeds from (5.12).

Theorem 3 is completely proven.

Let us supplement the assertions of theorem 3 by a series of remarks.

Remarks. 1. Under the conditions of theorem 3 the rate of convergence is characterized by the relationship

$$\|x_n - x_0\| = (1 + \gamma_n)\|[I - T'(x_0)]^{-1}(P^{(n)}x_0 - S_nx_n)\| \qquad (\gamma_n \to 0 \quad n \to \infty).$$

This result proceeds from (5.12) and (5.2).

The next remark will be used in ¶ 8.

2. Under the conditions of theorem 3 the following bilateral estimate is valid:

$$c_1\|P_nTx_0 - T_nP_nx_0\| \leqslant \|x_n - P_nx_0\| \leqslant c_2\|P_nTx_0 - T_nP_nx_0\|, \tag{5.13}$$

where $T_n = P_nT + S_n$.

For the sake of proof let us subtract from equality $x_n = P_nTx_n + S_nx_n$ the equality $P_nx_0 = P_nTP_nx_0 + (P_nTx_0 - P_nTP_nx_0)$, term by term. By making simple transformations, we obtain

$$[I - P_nT'(P_nx_0)]^{-1}(x_n - P_nx_0) = -z_n + P_n\omega(P_nx_0;\ x_n - P_nx_0), \tag{5.14}$$

where $z_n = P_nTx_0 - P_nTP_nx_0 - S_nx_n$. Analogously to (5.11) it is determined that

$$\frac{\|P_n\omega(P_n\ x_0;\ x_n - P_nx_0)\|}{\|x_n - P_nx_0\|} \to 0 \quad \text{when} \quad n \to \infty.$$

Jointly with (5.14) this has as a consequence the estimate /734

$$c'\|z_n\| \leqslant \|x_n - P_nx_0\| \leqslant c''\|z_n\|. \tag{5.13'}$$

But $z_n = (P_nTx_0 - T_nP_nx_0) + (S_nP_nx_0 - S_nx_n)$ and

$$\|S_nP_nx_0 - S_nx_n\| \leqslant \|x_n - P_nx_0\| \sup_{0<\theta_n<1} \|S_n'(x_n) - S_n'(x_n + \theta_n(P_nx_0 - x_n))\| = o(\|x_n - P_nx_0\|),$$

therefore (5.13') is equivalent to (5.13).

3. Under the conditions of theorem 3 is valid the estimate $(c = \text{const})$

$$\|x_n - x_n^0\| \leqslant c\|S_nx_n^0\|, \tag{5.15}$$

where x_n and x_n^0 are the solutions of equations (2.2) and (5.7), respectively.

Actually, the existence and uniqueness of solutions x_n and x_n^0 has been determined in the proof of theorem 3; it was noted that $\|S_nx_n^0\| \to 0$ when $n \to \infty$. To derive at estimate (5.15) let us write inequality (5.9) with the smallest possible constant δ i.e., we shall assume

$$\delta = \delta_n = \frac{\varkappa}{1-q}\|S_nx_n^0\|$$

and thereby shall convert inequality (5.9) into an equality. It is clear that in equality (5.8) remains also in force when $\delta = \delta_n$. Again applying the already used analog of theorem 2', we conclude that with sufficiently large values of n equation (2.2) has in the sphere $\|x - x_n^0\| \leqslant \delta_n$ a unique solution. By virtue of the uniqueness in the sphere $\|x_n - x_n^0\| \leqslant \delta_n$, (see theorem 3), this solution coincides with x_n. Consequently $\|x - x_0\| \leqslant \delta$ and this is estimate (5.15)

The following propositions are proved by means of analogous considerations.

4. Let the conditions of theorem 3 be satisfied when $S_n = S_n^{(i)}$ $(i = 1, 2)$. Then

$$\|x_n^{(1)} - x_n^{(2)}\| \leqslant c\|(S_n^{(1)} - S_n^{(2)})x_n^{(1)}\|, \quad \|x_n^{(1)} - x_n^{(2)}\| \leqslant c\|(S_n^{(1)} - S_n^{(2)})x_n^{(2)}\|.$$

where $x_n^{(i)}$ is the solution of equation (2.2) when $S_n = S_n^{(i)}$ $(i = 1, 2)$.

Remarks 3 and 4 allow investigation of the stability of various approximate methods with respect to comparatively minor errors in the equation system of the corresponding methods.

In proving estimate (5.1), we have essentially determined the following result.

Lemma 1. Let x_0 and x_n be the solutions, respectively, of equations (2.1) and (2.2) correspondingly $\|x_n - x_0\| \to 0$ when $n \to \infty$. Let the operators P_nT $(n = 1, 2, \ldots)$ be differentiable according to Frechet in point x_0,

$$P_nT(x_0 + h) - P_nTx_0 = (P_nT)'(x_0)h + \omega_n(x_0; h)$$

and furthermore:

$$1) \quad \frac{\|\omega_n(x_0; x_n - x_0)\|}{\|x_n - x_0\|} \to 0 \text{ when } n \to \infty;$$

2) operators $I - (P_nT)'(x_0)$ are continuously invertible and norms

$$\|I - (P_nT)'(x_0)\|, \ \|[I - (P_nT)'(x_0)]^{-1}\| \qquad (n \geqslant n_1)$$

are bounded in the set.

Then estimate (5.1) is valid.

Let us note that condition 1) is definitely satisfied in the tendency $\|\omega_n(x_0; h)\| / \|h\| \to 0$ when $\|h\| \to 0$ is uniform with respect to n.

The theorem 3 can be modified in several ways. We shall not devote time to this question. Let us merely note some more particular assertions which, however, are more useful for applications. /735

Theorem 4. Let operator T be continuous on open set $\Omega \subset E$, equation (2.1) have the solution $x_0 \in \Omega$ and the following conditions be satisfied:

1) operator T is continuously differentiable according to Frechet in point x_0 and linear operator $I - T'(x_0)$ is continuous invertible;

2) for any value of x from a vicinity $\|x - x_0\| < \sigma$ point x_0 and for value of $z \in E$ the inequality

$$\rho(T'(x)z, E_n) \leqslant \eta_n' \|z\| \qquad (n = 1, 2, \ldots);$$

is satisfied;

3) operators P_n $(n = 1, 2, \ldots)$ are bounded in E, and furthermore

$$\|P_n\|\eta_n' \to 0, \ \|P^{(n)}x_0\| \to 0 \text{ when } n \to \infty \text{ *)};$$

*) For the fulfillment of relation $\|P^{(n)}x_0\| \to 0$ it is sufficient that $\rho(x_0, E_n)\|P_n\| \to 0$ when $n \to \infty$.

4) operators S_n are continuous and differentiable according to Frechet in the set $\Omega_n = \Omega \cap E_n$, and furthermore

$$\sup_{x\in\Omega_n} \|S_n x\| \to 0, \qquad \sup_{x\in\Omega_n} \|S_n'(x)\| \to 0 \quad \text{when} \quad n\to\infty. \tag{5.16}$$

Then the conditions of theorem 3 are satisfied.

Verification is only required for condition 3) of theorem 3. Using conditions 2) and 3) of the theorem undergoing proof, we obtain when $\|x - x_0\| < \sigma$ and any $z \in E$,

$$\|P^{(n)}T'(x)z\| \leqslant \|P^{(n)}\|\, \rho(T'(x)z, E_n) \leqslant (1+\|P_n\|)\eta_n'\|z\|,$$

whence

$$\|P^{(n)}T'(x)\| \leqslant (1+\|P_n\|)\eta_n' \to 0 \quad \text{when} \quad n\to\infty \quad (\|x-x_0\|<\sigma).$$

Thus condition 3) of theorem 4) is satisfied.

Theorem 5. Let operator T be entirely continuous on open set $\Omega \subset E$ into a Banach space $E' \subset E$ (thus T is also entirely continuous as an operator in E), and projection operators P_n as operators from E' into E are bounded and tend strongly toward the operator of the insertions of E' into E. Let equation (2.1) have the solution $x_0 \in \Omega$ and let operator T be continuously differentiable according to Frechet in point x_0 as an operator from E in E' (in just the same manner as an operator in E), and furthermore let the homogeneous equation $h = T(x_0)h$ have on the trivial solution $h = 0$. Finally, let operators S_n ($n = 1, 2, \ldots$) be continuous and differentiable according to Frechet (as operators in E_n) on the set $\Omega_n = \Omega \cap E_n$ and let conditions (5.16) be satisfied.

Then the conditions of theorem 3 are satisfied.

Actually, $x_0 = Tx_0 \in E'$, and according to the condition of the theorem, $\|P^{(n)}z\|_E \to 0$ for any value of $z \in E'$. Therefore $\|P^{(n)}x_0\| \to 0$, and condition 1) of theorem 3 is satisfied. Furthermore, the Frechet derivative of an entorely continuous operator is an entirely continuous linear operator (see 4)). Thus $T'(x_0)$ is entirely continuous from E into E', and since $P_n \to P$ strongly (P is the insertion operator of E' in E), it follows that $\|P^{(n)}T'(x_0)\| \to 0$ when $n \to \infty$. We obtain /736

$$\|P^{(n)}T'(x)\| \leqslant \|P^{(n)}\|\,\|T'(x) - T'(x_0)\| + \|P^{(n)}T'(x_0)\|.$$

There $P^{(n)}T'(x)$ is considered as an operator in E, $P^{(n)} = P - P_n$ is considered as an operator from E' into E, and $T'(x)$ is considered as an operator from E into E'. According to the Banach - Steinhaus theorem norms, $\|P^{(n)}\|$ ($n = 1.2 \ldots$) are bounded in the sets, and according to the condition of theorem $\|T'(x) - T'(x_0)\| \to 0$ when $\|x - x_0\| \to 0$. Therefore, for any value of $\varepsilon > 0$, $\sigma_\varepsilon > 0$ and n_ε can be pointed out, such that

$$\|P^{(n)}T'(x)\| \leqslant \varepsilon \quad \text{when} \quad n \geqslant n_\varepsilon \quad \text{and} \quad \|x - x_0\| \leqslant \sigma_\varepsilon.$$

Thus, condition 3) of theorem 2 is satisfied; the remaining conditions of Theorem 2 are satisfied in an obvious manner.

Theorem 5 is valid, in particular, when E' = E; operators P_n in this case must tend strongly toward the single operator.

Let us finally point out a result, close to theorem 5, which follows from the corollary and remark 1 to theorem 1 and from lemma 1.

Theorem 5'. Let operator T be entirely continuous on an open set $\Omega \subset E$ as an operator from E into the Banach space E' E, continuously inserted into E, and let projection operators P_n, as operators from E' with E, be bounded and tend strongly toward the operator of this insertion of E' into E. Let equation (2.1) have the solution $x_0 \in \Omega$; and let operator T be differentiable according to Frechet in point x_0 as an operator from E into E', the homogeneous equation $h = T'(x_0)h$ having only a trivial solution*.) Finally let operators S_n (n = 1, 2, . . .) be entirely continuous in the set $\Omega_n = \Omega \cap E_n$ (as operators in E_n), with

$$\sup_{x \in \Omega_n} \|S_n x\| \to 0 \quad \text{when} \quad n \to \infty.$$

Then such values of n_0 and $\delta_0 > 0$, will be found, that the solution of equation (2.1) is unique in sphere $\|x - x_0\| \leqslant \delta_0$ and when $n \geqslant n_0$ equations (2.2) has in this sphere at least one solution x_n. Convergence $\|x_n - x_0\| \to 0$ occurs when $n \to \infty$ and evaluation (5.1) is valid.

The uniqueness of approximation x_n under the conditions of theorem 5' can be violated.

An analogous modification may be established for theorem 4.

¶6 . Combination with the Newton Method

Although equation (2.2) is solved more simply than is the initial equation (2.1), nevertheless the direct solution of this nonlinear equation can lead to great difficulties. Therefore it is of interest to consider the question of an approximate solution of equations (2.) themselves. In this section let us point out the possibility of utilizing Newton's method for this purpose. The utilization of Newton's method not to the initial equation (2.1) but to equations (2.2) is advantageous with respect to numerical realization. /737

For the duration of this section we assume that

$$E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots.$$

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* From this it follows (see (4)) that x_0 is an isolated solution of nonzero index. Let us note, that in distinction from theorem 5, we now do not assume this continuous differenciability of operator T.

We consider operators T, P_nT, and S_n to be differentiable (it is sufficient to satisfy the condition of theorem 3). To abbreviate the notation, let us introduce the designation $T_n = P_nT + S_n$. Operators T_n and $T_n'(x)$ $(x \in E_n)$ act in subspace E_n.

The simplest scheme of applying Newton's method to the solution of equations (2.2) consists in the following. At some value of $n = n_0$, there exists and initial approximation $x_{n_0,0}$ of the solution x_{n_0} of equation (2.2) Let us improve this initial approximation by performing in (2.2) one iteration according to Newton's method (see 2)). The obtained element $x_{n_0,1}$, is taken as the initial approximation $x_{n_0+1,0}$, of the solution of equation (2.2) when $n = n_0 + 1$, etc.:

$$x_{n,1} = x_{n,0} - [I - T_n'(x_{n,0})]^{-1}(x_{n,0} - T_n x_{n,0})$$
$$(x_{n,1} = x_{n+1,0}; \quad n = n_0, n_0 + 1, \ldots).$$

It would be natural to consider a more general iteration scheme, performing for each value of n, generally speaking. several iterations*):

$$x_{n,k+1} = x_{n,k} - [I - T_n'(x_{n,k})]^{-1}(x_{n,k} - T_n x_{n,k}) \tag{6.1}$$
$$(k = 0, \ldots, l_n; \quad l_n \geqslant 0; \quad x_{n,l_n+1} = x_{n+1,0}; \quad n = n_0, n_0 + 1, \ldots).$$

Theorem 6. Let the conditions of theorem 3 be satisfied.

Then if the initial value of $n = n_0$ is sufficiently large, and the initial approximation $x_{n_0,0}$ is sufficiently close to solution x_{n_0} of equation (2.2) when $n = n_0$, then iteration scheme (6.1) is realizable and $\|x_{n_0,0} - x_0\| \to 0$ when $n \to \infty$ (x_0 is the solution of equation (2.1)).

Proof. By virtue of (5.5) and convergence $\|x_n - x_0\| \to 0$ there exists a sufficiently small value of $\delta > 0$, such that

$$\|[I - T_n'(x)]^{-1}\| \leqslant \varkappa \qquad (\|x - x_n\| \leqslant \delta, \ x \in E_n; \quad n \geqslant n_0). \tag{6.2}$$

If necessary, by decreasing σ and increasing n_0, we can obtain result that

$$\sup_{\|x - x_n\| \leqslant 2\delta} \|T_n'(x) - T_n'(x_n)\| \leqslant \frac{1}{4\varkappa}. \tag{6.3}$$

Increasing n_0 once again if necessary, we shall consider that

$$\|x_n - x_m\| \leqslant \frac{\delta}{2} \text{ when } \quad m > n \geqslant n_0. \tag{6.4}$$

---

* At some values of n, it is possible not to iterate at all, taking x_{n,l_n+1} as an approximation of $x_{n+m+1,0}$ $(m > 0)$. To formally coordinate this jump with (6.1), it is necessary to exclude from sequence $\{E_n\}$ all subspaces between E_n and E_{n+m+1}, i.e., to redesignate E_{n+m+j} with E_{n+j} and analogously with T_{n+m+j} and $x_{n+m+i,k}$ (f = 1,2, ...).

This value n_0 we shall take as the initial one in iteration scheme (6.1). /738
Let us show that if the initial approximation $x_{n_0,0}$ belongs to sphere $\|x - x_{n_0}\| \leqslant \delta$, then iteration scheme (6.1) is realizable and leads to the converging process.

Let us assume that for certain values of n and k $(n \geqslant n_0, 0 \leqslant k \leqslant l_n)$ the approximation $x_{n,k}$ is defined and $\|x_{n,k} - x_n\| \leqslant \delta$ (this condition is satisfied, in particular, by the initial approximation $x_{n_0,0}$). Then operator $I - T_n'(x_{n,k})$ is invertible (see 6.2)) and, consequently, approximation $x_{n,k+1}$ exists. Preceeding from (6.1) and using the equality $x_n = T_n x_n$, we find

$$x_{n,k+1} - x_n = x_{n,k} - x_n - [I - T_n'(x_{n,k})]^{-1}(x_{n,k} - T_n x_{n,k}) =$$
$$= -[I - T_n'(x_{n,k})]^{-1}[T_n x_n - T_n x_{n,k} - T_n'(x_{n,k})(x_n - x_{n,k})].$$

Hence, taking into account (6.2) and inequality (4.3) for operator T_n, we obtain

$$\|x_{n,k+1} - x_n\| \leqslant \varkappa \|x_{n,k} - x_n\| \sup_{\|x - x_{n,k}\| \leqslant \delta} \|T_n'(x) - T_n'(x_{n,k})\|.$$

When $\|x - x_{n,k}\| \leqslant \delta$ we have $\|x - x_n\| \leqslant 2\delta$ and, in view of (6.3)

$$\sup_{\|x - x_{n,k}\| \leqslant \delta} \|T_n'(x) - T_n'(x_{n,k})\| \leqslant \sup_{\|x - x_n\| \leqslant 2\delta} \|T_n'(x) - T_n'(x_n)\| +$$
$$+ \|T_n'(x_n) - T_n'(x_{n,k})\| \leqslant \frac{1}{4\varkappa} + \frac{1}{4\varkappa} = \frac{1}{2\varkappa}.$$

consequently

$$\|x_{n,k+1} - x_n\| \leqslant \frac{1}{2} \|x_{n,k} - x_n\| \leqslant \frac{1}{2}\delta. \tag{6.5}$$

We have convinced outselves, in particular, that $x_{n,k+1}$ also belongs to sphere $\|x - x_n\| \leqslant \delta$; consequently, with a fixed value of n it is possible to perform any number of iterations (6.1) with respect to k. These we shall have

$$\|x_{n,l_n+1} - x_n\| \leqslant \left(\frac{1}{2}\right)^{l_n+1} \delta \leqslant \frac{1}{2}\delta,$$

which, joint with (6.4) yields, (recollect that $x_{n,l_n+1} = x_{n+1,0}$)

$$\|x_{n+1,0} - x_{n+1}\| \leqslant \|x_{n,l_n+1} - x_n\| + \|x_n - x_{n+1}\| \leqslant \delta. \tag{6.6}$$

Thus, $x_{n+1,0}$ enters into sphere $\|x - x_{n+1}\| \leqslant \delta$ and, as a consequence of induction, iterations (6.1) are realization for all values of $n \geqslant n_0$.

Let us prove that $\|x_{n,0} - x_0\| \to 0$ when $n \to \infty$. According to what has been proven, we have (see (6.5) and (6.6)).

$$\|x_{j,0} - x_j\| \leqslant \|x_j - x_{j-1}\| + \frac{1}{2}\|x_{j-1,0} - x_{j-1}\| \qquad (j > n_0).$$

Iterating this estimate with respect to j, we find

$$\| x_{n_0+m,0} - x_{n_0+m} \| \leqslant \frac{\varepsilon_1}{2^{m-1}} + \frac{\varepsilon_2}{2^{m-2}} + \ldots + \varepsilon_m + \frac{\| x_{n_0,0} - x_{n_0} \|}{2^m},$$

when

$$\varepsilon_j = \|x_{n_0+j} - x_{n_0+j-1}\| \to 0 \quad \text{when} \quad j \to \infty.$$

It can easily be seen that from lim ε_m = 0 it follows that /739

$$\lim_{m\to\infty} \left(\frac{\varepsilon_1}{2^{m-1}} + \frac{\varepsilon_2}{2^{m-2}} + \ldots + \varepsilon_m \right) = 0,$$

and in our case this signifies that $\|x_{n,0} - x_n\| \to 0$ when $n \to \infty$. Since $\|x_n - x_0\| \to 0$ when $n \to \infty$, it also follows that $\|x_{n,0} - x_0\| \to 0$ when $n \to \infty$.

Theorem 6 has been proven.

It is possible to combine (2.2) with the modified Newton method in various ways. The theorem 6 and its proof can be transferred almost word for word to the iteration scheme:

$$x_{n,k+1} = x_{n,k} - [I - T_n'(x_{n,0})]^{-1}(x_{n,k} - T_n x_{n,k})$$
$$(k = 0, \ldots, l_n, \quad l_n \geqslant 0; \; x_{n,l_n+1} = x_{n+1,0}; \quad n = n_0, n_0 + 1, \ldots).$$

Here operator I - Tn'(x) is inverted at each value of n only in the one point $x_{n,0}$, and several iterations are performed with respect to k. Of great interest are ineration schemes in which operator I - Tn'(x) is inverted only for an initial value of n = n_0, since as n increases, the difficulty of inverting this operator rapidly increases. Let us note some results in this direction; The proofs are analogous to the proof of theorem 6.

Theorem 6. Let the conditions of theorem 3 be satisfied.

Then if the initial value of n_0 is sufficiently large, and the initial approximation $x_{n_0;0}$ is sufficiently close to solution x_{n_0} of the equation (2.2) when n = n_0, then the iteration scheme

$$x_{n,k+1} = x_{n,k} - \Gamma_0(x_{n,k} - T_n x_{n,k})$$
$$(k = 0, \ldots, l_n, \quad l_n \geqslant 0; \quad x_{n,l_n+1} = x_{n+1,0}; \quad n = n_0, n_0 + 1, \ldots)$$

is realizable and leads to a convergent process (i.e., $\|x_{n,0} - x_0\| \to 0$ when $n \to \infty$) in any of the following three cases:

1) $\Gamma_0 = [I - P_{n_0} T'(x_{n_0,0})]^{-1}$;

2) $\Gamma_0 = [I - P_{n_0} T'(x_{n_0,0}) - S_{n_0}'(x_{n_0,0}) P_{n_0}]^{-1}$, operator P_n is bounded in E and $\|P_n\| s_n' \to 0$ when $n \to \infty$, where $s_n' = \sup_{x \in \Omega_n} \|S_n'(x)\|$;

3) $\Gamma_0 = [I - T_{n_0}'(x_{n_0,0}) P_{n_0}]^{-1}$, operators P_n are bounded in E and tend strongly toward a single operator, $\|T'(x_0) P^{(n)}\| \to 0$ when $n \to \infty$.

Great interest is afforded by effective conditions for the initial value of n_0 and x_{n_0},o, which guarantee the convergence of the various iteration schemes. With definite assumptions about the smoothness of operators T and T_n, such conditions may be derived on the basis of the results of (5).

7. A posteriori estimate of error

The basis in the determination of a posteriori error estimates is the corollary of theorem 2. Let us note that if the conditions of theorem 3 are satisfied, then this corollary permits us in principle to obtain asymptomatically precise estimates for norms $\|x_n - x_0\|$, i.e., such estimates $\|x_n - x_0\| \leqslant \beta_n$, that $\|x_n - x_0\| / \beta_n \to 1$ when $n \to \infty$. Actually, the norms $\|[I - T'(x_n)]^{-1}$ ($n \geqslant n_1$) under the conditions of theorem 3 are bounded as a set (see 5.3)) and, by assigning an arbitrarily small value to $q (0 < q < 1)$, it is possible to find such values of $\sigma > 0$ and n_q, that values $x_* = x_n$ and $n \geqslant n_q$ inequalities (4.8) and (4.9) hold true, and together with them estimate (4.10) also holds true. In view of arbitrariness of q, this signifies that estimate (4.10) is asymptomatically precise. /740

The practical application of estimate (4.10) is possible if the estimate of the norm $\|[I - T'(x_n)]^{-1}\|$, is known. Estimation of this norm represents, generally speaking, a difficult problem. One possible, and the most natural way of overcoming this difficulty consists in reducing this problem to the more simple problem of estimating norm $\|[I - P_nT'(x_n) - S_n'(x_n)]^{-1}\|$ в E_n; the latter norm can frequently be estimated in the process of finding solution x_n of equation (2.2)

Lemma 2. Let the operators T, P_nT and S_n be differentiable in point $x_* \in \Omega_n$, $(P_nT)'(x_*) = P_nT'(x_*)$, operator I - $P_nT'(x_*)$ - $S_n'(x_*)$ being invertible in E_n,

$$\|[I - P_nT'(x_*) - S_n'(x_*)]^{-1}\| \leqslant \varkappa_n. \tag{7.1}$$

Then if

$$q' \equiv (1 + \varkappa_n \|P_nT'(x_*)\|)\|P^{(n)}T'(x_*)\| + \varkappa_n \|S_n'(x_*)\| < 1, \tag{7.2}$$

it follows that operator I - $T'(x_*)$ is invertible in E and

$$\|[I - T'(x_*)]^{-1}\| \leqslant \frac{1 + \varkappa_n \|P_nT'(x_*)\|}{1 - q'}. \tag{7.3}$$

---

* An analogous assertion may be obtained by applying theorem 4(1.XIV) from (2), if operator I - $T'(x_*)$ possesses the following property: from the existance of a left-hand inverse it follows that there exists a bilateral inverse.

Proof*) of the lemma is based upon the following simple facts (see (2)): if A is a linear invertible operator in Banach space F, and linear operator B satisfies the condition $\|B\| \, \|A^{-1}\| < 1$, then operator A + B also invertible and

$$\|(A+B)^{-1}\| \leqslant \frac{\|A^{-1}\|}{1-\|B\| \, \|A^{-1}\|}.$$

Applying this proposition when $F = E_n$, $A = I - P_nT'(x_*)$, $B = S_n'(x_*)$, we obtain by virtue of conditions (7.1) and (7.2) the result that operator $I - P_nT'(x_*)$ is invertible in E_n and

$$\|[I-P_nT'(x_*)]^{-1}\|_{E_n} \leqslant \frac{\varkappa_n}{1-\varkappa_n \|S_n'(x_*)\|}. \tag{7.4}$$

We have

$$[I-P_nT'(x_*)]^{-1} = I + [I-P_nT'(x_*)]^{-1} P_nT'(x_*). \tag{7.5}$$

The operator in the right-hand part of (7.5) makes sense for any value of $x \in E$, since before $[I - P_nT'(x_*)]^{-1}$ stands the operator P_n of projection into E_n. Direct calculations show that the operator in the right-hand part of (7.5) is the inverse of $I - P_nT'(x_*)$ in the entire space E. Thus, operator $I - P_nT'(x_*)$ is invertible in E, and by virtue of (7.4) and (7.5)

$$\|[I-P_nT'(x_*)]^{-1}\|_E \leqslant 1 + \frac{\varkappa_n \|P_nT'(x_*)\|}{1-\varkappa_n\|S_n'(x_*)\|} \leqslant \frac{1+\varkappa_n\|P_nT'(x_*)\|}{1-\varkappa_n\|S_n'(x_*)\|}. \tag{7.6}$$

Applying the assertion formulated at the beginning of the proof of the lemma, when $F = E$, $A = I - P_nT'(x_*)$, $B = -P^{(n)}T'(x_*)$ by virtue of conditions (7.6) and (7.2) we arrive at estimate (7.3). Lemma 2 has been proven. /741

Theorem 7. Let operator T, P_nT and S_n be differentiable according to Frechet in a vicinity*) of point $\tilde{x}_n \in \Omega_n$, operator $I - P_nT'(\tilde{x}_n) - S_n'(\tilde{x}_n)$ being continuously invertible in E_n. Let

$$\varkappa_n = \|[I - P_nT'(\tilde{x}_n) - S_n'(\tilde{x}_n)]^{-1}\| \tag{7.7}$$

and

$$q_n' \equiv (1+\varkappa_n\|P_nT'(\tilde{x}_n)\|)\,\|P^{(n)}T'(\tilde{x}_n)\| + \varkappa_n\|S_n'(\tilde{x}_n)\| < 1,$$

and for certain values of δ_n and q_n $(\delta_n > 0,\ 0 \leqslant q_n < 1)$ the inequalities

$$\sup_{\|x-\tilde{x}_n\|\leqslant\delta_n} \|T'(x) - T'(\tilde{x}_n)\| \leqslant \frac{q_n}{\varkappa_n'}, \tag{7.8}$$

---

*) It is sufficient for P_nT and S_n to be differentiable only in the point $\tilde{x}_n$, S_n as an operator in E_n, and for P_nT to be differentiable as an operator in E, $(P_nT)'(\tilde{x}_n) = P_nT'(\tilde{x}_n)$.

**) It is assumed that sphere $\|x-\tilde{x}_n\| \leqslant \delta_n$ is contained in Ω.

$$\| \tilde{x}_n - T\tilde{x}_n \| \leqslant \frac{\delta_n (1-q_n)}{\varkappa_n'}, \tag{7.9}$$

where

$$\varkappa_n' = \frac{1+\varkappa_n \| P_n T'(\tilde{x}_n) \|}{1-q_n'}.$$

are satisfied.

Then equation (2.1) has the solution $x_0 \in \Omega$ and the error estimate

$$\frac{\alpha_n}{1+q_n} \leqslant \| \tilde{x}_n - x_0 \| \leqslant \frac{\alpha_n}{1-q_n}, \tag{7.10}$$

where

$$\alpha_n \equiv \| [I - T'(\tilde{x}_n)]^{-1} (\tilde{x}_n - T\tilde{x}_n) \| \leqslant \varkappa_n' \| \tilde{x}_n - T\tilde{x}_n \|. \tag{7.11}$$

is valid.

If conditions of theorem 3 are satisfied and $\|\tilde{x}_n - x_0\| \to 0$ when $n \to \infty$, then such values q_n, which satisfy the given conditions, can be so selected that $q_n \to 0$ when $n \to \infty$ and the estimate (7.10) will be asymptomatically exact.

Proof. The solubility of equation (2.1), estimate (7.10), and inequality (7.11) for α_n proceed directly from the corollary of Theorem 2 and from lemma 2. If the conditions of theorem 3 are satisfied and $\|\tilde{x}_n - x_0\| \to 0$, the norms (7.7) are bounded as a set (see (5.5)). Using condition 3) and 4) of theorem 3, we become convinced that $q_n' \to 0$ when $n \to \infty$ and consequently, values $\varkappa_n'$ are also bounded as a set. Let us take an arbitrarily sinale number $q > 0$. By virtue of correlationship $\|\tilde{x}_n - x_0\| \to 0$, $\|\tilde{x}_n - T\tilde{x}_n\| \to 0$ and condition 2) of theorem 3, there will also be found such values of $\delta > 0$ and n_q, that inequalities (7.8) and (7.9) will occur when $\delta_n = \delta$, $qn = q$ and $n \geqslant n_g$. And this means that q_n and δ_n may, in (7.8) and (7.9) be also related that $qn \to 0$ when $n \to \infty$. Theorem 7 is proven.

¶ 8. Application to integral equations. /742

Let us consider the nonlinear integral equation

$$x(t) = \int_a^b K(t, s, x(s))\, ds \tag{8.1}$$

with a continuous kernel K(t, s, u). Proceeding from the quadrature formula

$$\int_a^b z(s)\, ds = \sum_{j=1}^n a_{jn} z(s_{jn}) + R_n(z) \quad (a \leqslant s_{1n} < s_{2n} < \ldots < s_{nn} \leqslant b), \tag{8.2}$$

We shall instead of solution $x_0(t)$ of the equation (8.1), seek its value in the points of interpolation s_{jn}. We determine approximate values $\xi_{jn} \approx x_0(s_{jn})$ equation from the system of

$$\xi_{in} = \sum_{j=1}^{n} \alpha_{jn} K(s_{in}, s_{jn}, \xi_{jn}) \qquad (i = 1, \ldots, n). \tag{8.3}$$

This system is obtained from (8.1) if the integral of K(t, s, x(s)) is replaced in accordance with quadrature formula (8.2) (in which the residual term $R_n(z)$ is disregarded), and then if to variable t are successively given the values $s_{jn}, \ldots, s_{nn}$.

Theorem 8. Let the coefficients α_{jn} (j = 1, ...,n) of quadrature formula (8.2) be positive, for each value of n, and let quadrature process (8.2) converge, i.e., for any function z(s) which is continuous on segment $[a, b]$ let the relationship

$$R_n(z) = \int_a^b z(s)\,ds - \sum_{j=1}^{n} \alpha_{jn} z(s_{jn}) \to 0 \quad \text{when} \quad n \to \infty. \tag{8.4}$$

be satisfied.

Then if equation (8.1) has an isolated solution $x_0(t)$ with a nonzero index and kernel K(t, s, u) is continuous for a set of variables in the domain

$$a \leqslant t, \quad s \leqslant b, \quad x_0(s) - \delta \leqslant u \leqslant x_0(s) + \delta \quad (\delta > 0), \tag{8.5}$$

then with sufficiently large values of n equation system (8.3) has the solution $\xi_{1n}, \ldots, \xi_{nn}$

$$\max_{1 \leqslant j \leqslant n} |\xi_{jn} - x_0(s_{jn})| \to 0 \quad \text{when} \quad n \to \infty. \tag{8.6}$$

If in domain (8.5) there exists a continuous partial derivative J(t, s, u)/ u and the linear integral equation

$$h(t) = \int_a^b \frac{\partial K(t, s, x_0(s))}{\partial u} h(s)\,ds \tag{8.7}$$

has only the trivial solution $h(t) \equiv 0$ *), then the solution of system (8.3) is locally unique: such a value $\delta_0 > 0$, independent of n, is found with sufficiently large values of that n system (8.3) has a single solution which satisfies the condition /743

$$\max_{1 \leqslant j \leqslant n} |\xi_{jn} - x_0(s_{jn})| \leqslant \delta_0.$$

The rate of convergence is characterized by the inequalities $(c_1, c_2 = \text{const} > 0)$

$$c_1 r_n \leqslant \max_{1 \leqslant j \leqslant n} |\xi_{jn} - x_0(s_{jn})| \leqslant c_2 r_n, \tag{8.8}$$

---

* From this it already follows that the index of solution $x_0(t)$ it differs from zero.

where

$$r_n = \max_{1\leqslant i\leqslant n} |R_n(z_{in})|, \qquad z_{in}(s) = K(s_{in}, s, x_0(s)). \tag{8.9}$$

Several same complements and generalizations are indicated in the concluding part of the section.

Let us preface the proof of the theorem with one auxiliary result. Let us adopt the prepositions of the theorem concerning the positiveness of α_{jn} and convergence of quadrature process (8.2). When $z(s) \equiv 1$, it follows from (8.4) that

$$\sum_{j=1}^{n} \alpha_{jn} = (b-a) + \gamma_n, \qquad \gamma_n \to 0 \text{ when } n \to \infty.$$

Let $\beta_{jn} = \dfrac{b-a}{b-a+\gamma_n} \alpha_{jn}$ and

$$\gamma_{jn} = \alpha_{jn} - \beta_{jn} \qquad (j = 1, \ldots, n). \tag{8.10}$$

It is obvious that $\beta_{jn} > 0$, $\sum_{j=1}^{n} \beta_{jn} = b-a$ when $n \to \infty$ and

$$\sum_{j=1}^{n} |\gamma_{jn}| = |\gamma_n| \to 0 \tag{8.11}$$

(all values of γ_{jn} have the same sign as γ_n).

Let us divide segment $[a, b]$ by points

$$a, a+\beta_{1n}, a+\beta_{1n}+\beta_{2n}, \ldots, a+\sum_{j=1}^{n} \beta_{jn} = b$$

into n nonintersecting intervals

$$J_{1n} = [a, a+\beta_{1n}), \qquad J_{2n} = [a+\beta_{1n},\ a+\beta_{1n}+\beta_{2n}), \ldots,$$
$$J_{nn} = [b-\beta_{nn},\ b].$$

Let us introduce subjects $D_{jn} \subset [a, b]$ $(j = 1, \ldots, n)$ according to the following rule. If $s_{jn} \in J_{jn}$ $(j = 1, \ldots, n)$, then we simply assume $D_{jn} = J_{jn}$ $(j = 1, \ldots, n)$. In the contrary case we transfer point s_{jn} to J_{jn}:

$$D_{jn} = (J_{jn} \setminus \bigcup_{i=1}^{n} s_{in}) \cup s_{jn} \qquad (j = 1, \ldots, n)$$

(in this notation s_{jn} is regarded as a one-element set consisting of the number s_{jn}). It is clear that

$$s_{jn} \in D_{jn}, \qquad \bigcup_{j=1}^{n} D_{jn} = [a, b], \qquad D_{in} \cap D_{jn} = \varnothing \quad \text{when} \quad i \neq j, \tag{8.12}$$

$$\operatorname{mes} D_{jn} = \operatorname{mes} J_{jn} = \beta_{jn} \qquad (j = 1, \ldots, n).$$

Our purpose is to show that the diameter of subset Djn tend toward zero: /744

$$d_n \equiv \max_{1 \leqslant j \leqslant n} \sup_{s', s'' \in D_{jn}} |s' - s''| \to 0 \text{ when } n \to \infty. \tag{8.13}$$

Relationship (8.13) can be violated for two reasons: either the lengths of intervals Jjn do tend toward zero when $n \to \infty$, or some points sjn are situated far from the corresponding values of Jjn. Let us show that in both cases quadrature process (8.2) does not converge, in spite of the condition.

The length of interval Jjn is equal to jn, and in the first case such a sequence $\beta_{j_k n_k}$, can be found that

$$\beta_{j_k n_k} \geqslant \eta \qquad (k = 1, 2, \ldots), \tag{8.14}$$

where η is a positive constant. Switching, if necessary, to a subsequence, one can be considered that the corresponding sequence of points of interpolation converges:

$$s_{j_k n_k} \to s_0 \in [a, b] \quad \text{when } n \to \infty.$$

For a function*) z(s) which is continuous on $[a, b]$, equal to 1 when $|s - s_0| \leqslant \eta/4$, equal to 0 when $\eta/4 \leqslant |s - s_0| \leqslant \eta/2$, and linear when $|s - s_0| \geqslant \eta/2$, we nave

$$\int_a^b z(s)\, ds = {}^3/_4\eta. \tag{8.15}$$

On the other hand, with sufficiently large values of k the point $s_{j_k n_k}$ gets into segment $|s - s_0| \leqslant \eta/4$, in which z(s) = 1, therefore

$$\sum_{j=1}^{n_k} \alpha_{j n_k} z(s_{j n_k}) \geqslant \alpha_{j_k n_k};$$

using conditions (8.11) and (8.14) we get with large values of k

$$\sum_{j=1}^{n_k} \alpha_{j n_k} z(s_{j n_k}) \geqslant \beta_{j_k n_k} - \frac{\eta}{8} \geqslant {}^7/_8\eta.$$

Comparing this with (8.15), we come to the conclusion, that quarantine process (8.2) does not converge for the constructed function.

In the second case, there can be found a sequence $s_{j_k n_k}$ of points of interpolation, the distance from which to the corresponding intervals $J_{j_k n_k}$ exceeds a number $\eta > 0$. Proceeding, if necessary, to a subsequence, let us consider that every point $s_{j_k n_k}$ is situated on the same side of corresponding value of $J_{j_k n_k}$ (for the sake of definiteness, on the right-hand side) and that the right-hand ends of intervals $J_{j_k n_k}$ tend toward $k \to \infty$ to a certain limit $s_1 \in [a, b]$.

* Here is investigated a case where $s_0 \in (a, b)$; the number $\eta > 0$ may then be considered so small that s_0 enters into $[a, b]$ together with the vicinity $(s_0 - \eta, s_0 + \eta)$. In a case where $s_0 = a$ or $s_0 = b$, function z(s) is constructed analogously.

The right-hand end of interval $J_{j_k n_k}$ has this form of $a+\sum_{j=1}^{j_k}\beta_{jn_k}$. Therefore /745

$$\sum_{j=1}^{j_k}\beta_{jn_k}\to s_1-a \text{ when } k\to\infty,$$

and in view of (8.11), also

$$\sum_{j=1}^{j_k}\alpha_{jn_k}\to s_1-a \text{ when } k\to\infty. \quad (8.16)$$

For a continuous function z(s), equal to 1 when $a\leqslant s\leqslant s_1$, equal to 0 when $s_1+\eta/2\leqslant s\leqslant b$ and linear when $s_1\leqslant s\leqslant s_1+\eta/2$, we have

$$\int_a^b z(s)\,ds=(s_1-a)+\frac{\eta}{4}. \quad (8.17)$$

On the other hand,

$$\sum_{j=1}^{n_k}\alpha_{jn_k}z(s_{jn_k})\leqslant\sum_{j=1}^{j_k}\alpha_{jn_k}+\sum_{j=j_k+1}^{n_k}\alpha_{jn_k}z(s_{jn_k}).$$

With sufficiently large values of k, the second sum in the right-hand part of the inequality vanishes. Actually, since point $s_{j_k n_k}$ is located to the right of $J_{j_k n_k}$ at a distance of $\geqslant\eta$, and the right-hand ends $J_{j_k n_k}$ tend toward s_1, it follows that $s_{j_k n_k}>s_1+\eta/2$ with large values of k, and moreover $s_{jn_k}>s_{j_k n_k}>s_1+\eta/2$ when $j>j_k$ and the same values of k, i.e., s_{jn_k} gets into the domain where z(s) = 0. Thus, with large values of k we have

$$\sum_{j=1}^{n_k}\alpha_{jn_k}z(s_{jn_k})\leqslant\sum_{j=1}^{j_k}\alpha_{jn_k}.$$

Comparing this with (8.16) and (8.17), we see that quadrature process (8.2) does not converge for the constructed function.

The proof of relationship (8.13) is completed.

Proof of theorem 8. We shall regard equation (8.1) as operator equation (2.1) in Banach space E of functions $\|x\|=\sup_{a\leqslant s\leqslant b}|x(s)|$ bounded and measured on segment $[a,b]$.

The continuity of kernel K(t, s, u) in domain (8.5) has a as a consequence the complete continuity of operator

$$Tx=\int_a^b K(t,s,x(s))\,ds$$

on sphere $\bar{\Omega}(\|x - x_0\| \leq \delta)$ as an operator from space E into space C of functions that are continuous on [a, b]. Moreover, T is completely continuous on sphere $\bar{\Omega}$ as an operator *) in E.

We shall demote by $\chi_{jn}(s)$ the characteristic function of the above-constructed set D_{jn}: /746

$$\chi_{jn}(s) = \begin{cases} 1 & \text{when } s \in D_{jn} \\ 0 & \text{when } s \notin D_{jn}. \end{cases} \quad (j = 1, \ldots, n)$$

Let E_n be the linear envelope of functions χ_{jn} (j = 1,..., n). Obviously E_n is a closed subspace in E. Let us introduce into E_n the operator T_n, determined on the basis of $\bar{\Omega}_n = \bar{\Omega} \cap E_n$ and acting according to the formula

$$T_n z_n = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} a_{in} K(s_{in}, s_{jn}, \zeta_j) \right\} \chi_{in} \qquad \left(z_n = \sum_{j=1}^{n} \zeta_j \chi_{jn} \in \bar{\Omega}_n \right)$$

From the linear independence of functions χ_{in} (i = 1, ..., n) there follows the equivalence of system (8.3) to the operator equation

$$x_n = T_n x_n \tag{8.18}$$

in E_n. The equivalence is understood in the following sense: vector $(\xi_{in}, \ldots, \xi_{nn})$ will be the solution of system (8.3) when and only when element

$$x_n = \sum_{j=1}^{n} \xi_{jn} \chi_{jn}$$

will be the solution of equation (8.18).

Let us furthermore introduce the linear operator P_n, which assigns to each function $x(s) \in E$ the function

$$P_n x = \sum_{j=1}^{n} x(s_{jn}) \chi_{jn}.$$

From (8.12) it follows that $P_n(E) = E_n$ and $P_n z_n = z_n$ for $z_n \in E_n$, i.e. P_n is a linear operation of projection for subspace E_n; P_n is bounded $\|P_n\| = 1$. Furthermore, for any function x(s) which is contiuous on [a,b] we have

$$\max_{a \leq s \leq b} \left| x(s) - \sum_{j=1}^{n} x(s_{jn}) \chi_{jn}(s) \right| \to 0 \text{ when } n \to \infty,$$

since according to what has been proven (see (8.13)), when $n \to \infty$, the diameters of subset D_{jn} tend toward zero, and $s_{jn} \in D_{jn}$. In other words, the sequence of operators P_n tends strongly toward the operator of the intersection of space C into space E.

We shall write equation (8.18) in the form of the perturbed Galerkin method:

*) The index of the solution $x_0(t)$ does not depend upon space E or C, in which equation (8.1) is investigated.

$$x_n = P_n T x_n + S_n x_n \qquad (S_n = T_n - P_n T). \qquad (8.19)$$

Operator S_n is continuous and in view of the finite dimensionality of E_n, is also completely continuous on $\bar{\Omega}_n$. For $z_n = \sum_{j=1}^{n} \zeta_j \chi_{jn} \in \bar{\Omega}_n$ we have /747

$$S_n z_n = T_n z_n - P_n T z_n = \sum_{i=1}^{n} \Big\{ \sum_{j=1}^{n} \alpha_{jn} K(s_{in}, s_{jn}, \zeta_j) -$$

$$- \int_a^b K(s_{in}, s, z_n(s))\, ds \Big\} \chi_{in} = \sum_{i=1}^{n} \Big\{ \sum_{j=1}^{n} \int_{D_{jn}} [K(s_{in}, s_{jn}, \zeta_j) - K(s_{in}, s, \zeta_j)]\, ds \Big\} \chi_{in} +$$

$$+ \sum_{i=1}^{n} \Big\{ \sum_{j=1}^{n} \gamma_{jn} K(s_{in}, s_{jn}, \zeta_j) \Big\} \chi_{in}$$

(we used relationships (8.10) and (8.12). Hence *)

$$\|S_n z_n\| \leqslant (b-a) \max_{1 \leqslant i,\, j \leqslant n} \sup_{s \in D_{jn}} |K(s_{in}, s_{jn}, \zeta_j) - K(s_{jn}, s, \zeta_j)| +$$

$$+ \sum_{k=1}^{n} |\gamma_{kn}| \max_{1 \leqslant i,\, j \leqslant n} |K(s_{in}, s_{jn}, \zeta_j)|;$$

as a consequence of (8.11) and (8.13) the boundaries and uniform continuity of function K(t,s,u) in the closed domain (8.5) we have

$$\sup_{x \in \bar{\Omega}_n} \|S_n x\| \to 0 \text{ when } n \to \infty.$$

We have proved that for the equations (8.1) and (8.19) the conditions of theorem 1 have been satisfied (see also the corollary and comment 1 to it **). In the indicated theorem equation (8.19) or, what is the same,

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*) Here and henceforth we make use of the obvious fact that for a function of the type $y_n = \sum_{i=1} \eta_i \chi_{in}$, there will be $\|y_n\| = \max_{1 \leqslant i \leqslant n} |\eta_i|$. In particular, for a function of the type

$$v_n = \sum_{i=1}^{n} \Big\{ \sum_{j=1}^{n} \int_{D_{jn}} v_{ij}(s)\, ds \Big\} \chi_{in}$$

we have

$$\|v_n\| \leqslant \max_{1 \leqslant i \leqslant n} \Big| \sum_{j=1}^{n} \int_{D_{jn}} v_{ij}(s)\, ds \Big| = \max_{1 \leqslant i,j \leqslant n} \sup_{s \in D_{jn}} |v_{ij}(s)| \sum_{k=1}^{n} \int_{D_{jn}} ds =$$

$$= (b-a) \max_{1 \leqslant i,j \leqslant n} \sup_{s \in D_{jn}} |v_{ij}(s)|.$$

**) The role of space E' continuously inserted into E is played by space C.

equation (8.18), at sufficiently large values of n have the solution x_n and $\|x_n - x_0\| \to 0$ when $n \to \infty$. But at the same values of n, according to what has been proved, equation system (8.3) is soluble, and from the inequality

$$\max_{1\leqslant j\leqslant n} |\xi_{jn} - x_0(s_{jn})| = \|x_n - P_n x_0\| \leqslant \|x_n - x_0\| \tag{8.20}$$

there follows the convergence (8.6). The first part of theorem 8 has been proved.

The existence of partial derivative $\partial K(t, s, u) / \partial u$ which is continuous in domain (8.5), leads to the continuous differentiability of operator T in point x_0 (as an operator from E into C and even more so as an oerator in E) and furthermore

$$T'(z)h = \int_a^b \frac{\partial K(t, s, z(s))}{\partial u} h(s)\, ds.$$

This same condition guarantees the differentiability of operator S_n while for $h_n = \sum_{j=1}^{n} \eta_j \chi_{jn} \in E_n$ and $z_n \in \overline{\Omega}_n$ we have

$$S_n'(z_n)h_n = \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \int_{D_{jn}} \left[\frac{\partial K(s_{in}, s_{jn}, \zeta_j)}{\partial u} - \frac{\partial K(s_{in}, s, \zeta_j)}{\partial u} \right] ds \eta_j \right\} \chi_{in} +$$

$$+ \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \gamma_{jn} \frac{\partial K(s_{in}, s_{jn}, \zeta_j)}{\partial u} \eta_j \right\} \chi_{in}.$$

Hence, utilizing the uniform continuity of $\partial K(t, s, u) / \partial u$ in the closed domain (8.5) as above, we find that /748

$$\sup_{x\in\overline{\Omega}_n} \|S_n'(x)\| \to 0 \text{ when } \to \infty.$$

Also taking into consideration the hypothesis of only a 0 solution (8.7) (equations h = $T'(x_0)h$), we can certify that all conditions of theorem 5 have been satisfied and even more so those of theorem 3. According to theorem 3, the solution x_n of equation (8.19) with sufficiently large values of n is unique in sphere $\|x - x_0\| \leqslant \delta_0$ of sufficiently small radius δ_0, and this is equivalent to confirmation of the theorem being proven concerning the local uniqueness of the solution of system (8.3).

Evaluation of (8.8) proceeds from (5.13) and equality (8.20) it is necessary only to note that $\|P_n T x_0 - T_n P_n x_0\| = r_n$, where r_n is a number determined by relationships (8.9).

Theorem 8 has been fully proven.

Remark. On the basis of solution $\xi_{in}, \ldots, \xi_{nn}$ of the system (8.3), an analytical approximation can be constituted by various methods. It is most convenient to assume

$$\bar{x}_n(t) = \sum_{j=1}^{n} a_{jn} K(t, s_{jn}, \xi_{jn}),$$

since compilation of this function does not require additional calculations. It can be easily seen that under the conditions of the second part of theorem 8, estimate

$$\max_{a \leq t \leq b} |\bar{x}_n(t) - x_0(t)| \leq c|R_n(z_t)|, \quad z_t(s) = K(t, s, x_0(s)).$$

is valid.

Theorem 8 can without difficulty be generalized into equation /749

$$x(t) = f\left(t, \int_a^b K_1(t, s, x(s))\, ds, \ldots, \int_a^b K_m(t, s, x(s))\, ds\right), \tag{8.21}$$

where $f(t, z_1, \ldots z_m)$ and $K_i(t,s,u)$ $(i = 1, \ldots, m)$ are continuous for the set of the variable function; an analog of system (8.3) has the form of

$$\xi_{in} = f\left(s_{in}, \sum_{j=1}^{n} \alpha_{jn} K(s_{in}, s_{jn}, \xi_{jn}), \ldots, \sum_{j=1}^{n} \alpha_{jn} K(s_{in}, s_{jn}, \xi_{jn})\right)$$
$$(i = 1, \ldots, n).$$

Boundary value problems for conventional differential equations may be reduced to equations of the type (8.21).

Let D be a closed domain in an m-dimensional Euclidian space, $t = (t_1, \ldots, t_m)$, $s = (s_1, \ldots, s_m)$. Let us assume that each node s_{jn} of "quadrature" formula

$$\int_D z(s)\, ds = \sum_{j=1}^{n} \alpha_{jn} z(s_{jn}) + R_n(z) \qquad (\alpha_{jn} > 0;\ s_{jn} \in D)$$

may be included into the measurable subset $D_{jn} \subset D$ in such a manner that conditions which are analogous to (8.12):

$$\bigcup_{j=1}^{n} D_{jn} = D, \quad D_{in} \cap D_{jn} = \varnothing \text{ when } i \neq j, \quad \operatorname{mes} D_{jn} = \left(\operatorname{mes} D \Big/ \sum_{j=1}^{n} \alpha_{jn}\right) \alpha_{jn},$$

would be satisfied, and furthermore this being the most important of all, subsets D_{jn} tend toward zero. Then theorem 8 and its proof are transferred, without substantial changes, to the equation

$$x(t) = \int_D K(t, s, x(s))\, ds.$$

The question remains open whether a subdivision of D into subsets D_{jn}, which possesses the enumerated properties, always exists for a converging quadrature process.

Let us finally indicate some bobliographical references. The convergence of the method of mechanical quadratures (8.3) for the nonlinear equation (8.1) in the case of Gauss interpolation formula was established in [6]. In [7] (also [8]) a posteriori error estimates have been derived in the case of an arbitrary quadrature formula; the question of convergence was not investigated- In the quoted works, continuity together with the first and second derivative with respect to u is required from kernel K(t,s,u). Theorem 8 apparently contains new results also in the case of a linear equation (the case of $K(t,s,u) = G(t,s)u + f(t)/(b - a)$). For a bibliography of works, concerning the method of mechanical quadrature for linear equations, see [9] and also [1], [2].

¶ 9. Application to More General Types of Equations /750

By means of substitution of the desired elements, the results of the previous paragraphs can be transferred to equations with unbounded operators. Let us consider the equality

$$Au = Bu \tag{9.1}$$

where A is a linear (generally speaking unbounded) operator and B is a nonlinear operator from Banach space $\mathfrak{E}$ into Banach space E. Let the domain of determination D(A) of operator A be dense in $\mathfrak{E}$, and let the domain of determination D(B) of nonlinear operator B be in an open subspace $\Delta \subset \mathfrak{E}$, and furthermore

$$D(A) \cap \Delta \subset D(B). \tag{9.2}$$

It is, in addition assumed that operator A is invertible and the inverse operator A^{-1} is bounded and determined on the entire space E.

In applications, cases often occur in which, although operator A^{-1} is bounded, it is determined only on a dense set in E. If in addition it is also known that A permits closure of $\bar{A}$, then operator $\bar{A}$ will already satisfy the established condition, i.e. $\bar{A}$ has a bounded inverse operator, defined on all of E. Therefore, we assume that the closure of operator A has already been effected .

It should be kept in mind that with the closure of operator A, inclusion (9.2) can be violated, and in such a case, simultaneously with the closure of A it is necessary somehow to expand operator B.

Let $\{E_n\}$ be a sequence of closed subspaces in E, let Pn be a linear operator of projection onto subspace E_n. Let us denote

$$\mathfrak{E}_n = A^{-1}(E_n) \qquad (\mathfrak{E}_n \subset \mathfrak{E};\quad n = 1, 2, \ldots).$$

As approximate solutions of equation (9.1) are adopted the solutions of equation

$$Au_n = P_n B u_n + C_n u_n, \tag{9.3}$$

where C_n is, generally speaking, a nonlinear operator from $\mathfrak{E}_n$ into E_n

determined on the set $\Delta_n = \Delta \cap \mathfrak{E}_n$. Operators A and P_nB in equation (9.3) also act from $\mathfrak{E}_n$ into E_n.

The substitution

$$x = Au \tag{9.4}$$

reduces equation (9.1) to the equation

$$x = Tx \qquad (T = BA^{-1}) \tag{9.5}$$

in space E, and reduces equations (9.3) to the equations

$$x_n = P_nTx_n + S_nx_n \qquad (T = BA^{-1}; \qquad S_n = C_nA^{-1}) \tag{9.6}$$

in subspace E_n. The domain of determination of operator $T = BA^{-1}$ is open in set E (see (9.2))

$$\Omega = (x : x \in E, \quad A^{-1}x \in \Delta),$$

and the domain of determination of operator $S_n = C_nA^{-1}$ is set $\Omega_n = \Omega \cap E_n$. Let the domain of determination of operator P_nT also be Ω, i.e. let set $T(\Omega)$ enter into the domain of determination of operator P_n.

After these remarks, the results of the preceding sections are directly transferred to equations (9.5) and (9.6), and at the same time also to equations (9.1) and (9.2), which are equivalent to them. We do not cite the formulations due to their obviousness. It can be shown that in the case of linear equations, substitution (9.4) leads to the same results as does the method, different in form, of reduction to the equations of the II kind, which was used in [1], [2]. /751

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REFERENCES

1. Kantorovich, L.V. Funktsional'nyy analiz i prikladnaya matematika (Functional analysis and applied mathematics), Uspekhi matem. nauk (Advances in math. sciences), vol.3, No.6(28), pp.89-185, 1948.
2. Kantorovich, L.V., Aplov, G.P. Funktsional'nyy analiz v normirovanykh prostranstvakh (Functional analysis in normalized spaces), Fizmatgiz, Moscow, 1959.
3. Vainikko, G. O skhodimosti kollokatsii dlya nelineynikh differentsial'nykh uravneniy (Concerning the convergence of the collocation method of nonlinear differential equations), Zhurnal vychislitel'noy matematiki i matematicheskoi fiziki (Journal of computer mathematics and math. physics), vol. 6, No.1, pp.35-42, 1966.
4. Krasnosel'skiy, M.A. Topologicheskiye metody v teorii nelineynikh integral'nykh uravneniy (Topological methods in the theory of nonlinear integral equations), Moscow, Gostekhizdat, 1956.

5. Krasnosel'skiy, M.A., Rutitskiy, Ya.B. O nekotorykh priblizhennykh metodakh resheniya nelineynikh operatornykh uravneniy, osnovannykh na linearizatsii.("Concerning certain approximate methods for solving nonlinear operator equations, based on linearization), Doklady AN SSSR (Report of Academy of Sciences, USSR),vol.141, No.4, pp.785-788, 1961.
6. Baluyev, A.N. O priblizhennom reshenii nelineynikh integral'nykh uravneniy (Concerning the approximate solution of nonlinear integral equations), Uch. zap. LGU (Scientif. notes of Leningrad State Univ.), issue 33, pp. 28-31, 1958.
7. Mysovskikh, I.M. O metode mekhanicheskikh kvadratur dlya resheniya integral'nykh uravneniy (Concerning the method of mechanical quadratures for solving integral equations), Vestn. LGU (Leningrad State Univ. Bulletin), No.7, pp.77-88, 1962.
8. Mysovskikh, I.M. Otsenka oshibki chislennogo resheniya nelineynogo integral'nogo uravneniya (Error estimate of the numerical solution of a nonlinear integral equation). Dokl. AN SSSR (Rep. Acad. Sci. USSR), vol.153, No.1, pp.30-33, 1963.
9. Kantorovich, L.V., Krylov, V.I. Priblizhennye metody vysshego analiza (Approximate methods of higher analysis). Moscow-Leningrad, Fizmatgiz, 1962.

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